Search in a Non-exclusive Online Matching Platform

(Preliminary and Incomplete)

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Abstract

Every year more people around the world use dating platforms. Yet little is understood about the marriage markets when there are profit maximizing platforms operating in the market. We analyze a standard search and matching model of a marriage market but with a profit maximizing platform. Subscribers to the platform enjoy an additional flow of contacts through the platform in exchange for a flow-price for subscription. We first describe the set of search equilibria when there is a platform. Then, we construct the demand function that the platform faces concentrating on equilibria without coordination failures. We find that market demand may locally behave like "Giffen goods" as there are ranges of prices where the market demand increases with price. Furthermore, we show as the search frictions by the platform vanishes, the matching among the subscribers does not necessarily converge to the stable matching.

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1 Introduction

Every year more people are using matchmaking platforms to search for marriage or dating partners. Matchmaking platforms such as Tinder accrue large sums of revenue, measured in billions of US dollars. However, the literature on matchmaking platforms in marriage markets is rather sparse. In this paper, we provide a search and matching model where individuals could subscribe to a monopolist platform to access to an additional search environment in exchange for a subscription price. Using this framework, we describe the equilibrium patterns of search with and without platform, and investigate the properties of the market demand as a function of the subscription price.

The literature on search and matching in marriage markets without platforms is rather well-developed. Notable examples of include, but are not limited to, Burdett and Coles (1997), Adachi (2003), Smith (2006), Lauermann and Nöldeke (2014), Lauermann et al. (2020) and Antler and Bachi (2022). In environments with homogeneous cardinal preferences, it is a well-established result that the whole society organizes around a recursive class-structure, where each class of types of individuals can expect to partner up with only among each other. To see this, suppose that there is a "star" in a marriage market, who is an acceptable marriage partner for everyone. However, for the star not everyone would be an acceptable partner. S/he can be as picky as s/he wants. Search frictions in the marriage market would determine how picky s/he would like to be. His/her acceptance strategy generates a cluster of individual types at the top that are willing to accept only each other but no one else. Then, given their behavior, the second-rate "stars" in the remaining population form a similar top-cluster. And this continues until the whole marriage market is partitioned into such clusters.

What happens if an online dating platform enters to the market? Now, the "stars" of the marriage market could subscribe to the platform to find each other faster in exchange for a subscription fee. The subscription fee and the search technology that the platform provides determines the search behavior of the subscribers. Unlike a typical consumption platform, the search behavior of the subscribers also affects the value of search through the platform and hence the value of subscription for other individuals. The implied coordination incentives among individuals both in their search behaviors and in subscription decisions could create an unusual market demand for subscription.

The literature on marriage markets with platforms is rather limited. In their seminal work, Bloch and Ryder (2000) investigate the pricing by a monopolist platform in a marriage market. In their model, the preferences over matches are determined by "pizzazz values" as in our model. In this environment, they assume that the platform implements the stable matching, which corresponds to the perfect sorting of types of individuals according to their pizzazz values. This type of matching can be interpreted as that the platform provides a search environment with no search frictions. Nonsubscribers search for partners outside with the underlying search frictions in the society. In a more recent work, Marx and Schummer (2021) consider more general preferences but do not allow for search outside. In both models, the subscription fee is a one-time payment conditional on matching through the platform, while in our model the subscription fee is only conditional on the time that a subscriber spends on the platform. In their empirical work, Hitsch et al. (2010) also employ a model where the platform implements a stable matching to estimate preferences of subscribers. Most recently, Antler et al. (2023) analyze online matching platforms but rather concentrate on the quality of matches implemented by platform in an environment with horizontal and vertical preferences over partners and subscription decisions are exogenous. In our model, we analyze the interaction between the subscription decisions and search behavior and therefore the relation between market demand for subscriptions and matching patterns.

When there are some search frictions, it is known that there is still some sorting in equilibrium but it is an imperfect one (Smith, 2006). When there is some search friction within the platform, the resulting matchings could be different from stable matchings. Furthermore, with a time-conditional subscription fee, the matching outcomes might depend on the pricing as subscription fee is an additional waiting cost for the subscribers. With this framework, we are able to investigate consumption externalities due to matching more explicitly and analyze welfare implications of a monopolist dating platform.

Our first main contribution is to define and describe the set of participation equilibria. A participation equilibrium is a search equilibrium but now agents can make a subscription choice which changes the search frictions they face. This addition of binary subscription choice create coordination incentives among the individuals. We find that the set of participation equilibria can be described with a binary tree instead of a single chain, as would be the case in a search equilibrium without platforms.

We construct a demand function by focusing on the participation equilibrium where there is no coordination failures among the individuals in their subscriptions decisions. That is, if subscription is beneficial for a group of people in case they expect each other to subscribe, we focus on the equilibrium where this group of people indeed subscribe. We show that it is possible to calculate the set of subscribers, the matching patterns, and the demand function by using a recursive algorithm.

Our first main result is that the market demand for subscription can increase with price. As the flow price increases, everyone who subscribes has a higher waiting cost. This makes them more "tolerant" for acceptance; that is, their acceptance sets expands. This motivates the top members from the next outsider cluster to subscribe as well because their subscription search value increases. As a result the equilibrium cluster that subscribes expands, which results in an increase in demand.

Finally, we show that endogenizing the unmatched type distribution is necessary to solve an "infinite profit paradox" that a model of exogenous unmatched type distribution might generate. If the platform could reduce search frictions indefinitely, it can also increase the price proportionally while keeping the demand function intact. To solve this problem, we turn to a more involved model with an exogenous flow of types of individuals to the population and a distinction between the exogenous population distribution and an endogenous unmatched type distribution. Following the formulation by Smith (2006), the unmatched type distribution is co-determined with the search equilibrium via a balance equation.

We first show that our results extend to this environment; although the existence proofs are more involved. Then, we show that the revenue of the platform does not explode as the search frictions vanish. It actually converges to a finite limit. With this result, we conjecture that there are environments where the platform does not have incentives to completely eliminate the search frictions.

Section 2 lays out the specifics of the marriage market and the search environment. Section 3 describes the environment with a platform.In Section 4 we construct the demand function, and finally in Section 5 we propose and analyze the full model.

2 Model

There is a set of men, $\mathcal{M}$, and a set of women, $\mathcal{W}$, each containing a unit mass of agents. Each agent is characterized by a number, which, following Burdett and Coles (1997), we refer to as the agent's pizzazz and assume to be distributed on the interval $[0, 1]$, according to an atomless continuous distribution F. We denote the corresponding density by f and refer to an agent with pizzazz a as agent a.

The market operates in continuous time. Each individual meets agents of the opposite sex at a flow rate μ_o, where μ_o is the parameter of a Poisson process. Meetings are random: agents meet agents of the opposite sex with pizzazz value in some measurable set M at a flow rate proportional to their mass in the population $\mu_o \int_M dF(x)$. When two agents meet, they immediately observe each other's pizzazz and decide whether to accept each other as a partner. If both agents accept, then they marry and exit the market. Otherwise, they return to the market and continue their search. When agent a marries agent b, the latter obtains a payoff of a and the former obtains a payoff of b. Agents obtain no flow payoff when single. Agents maximize their expected payoffs discounted at a rate $r > 0$.

When agents leave the market, they are immediately replaced by agents with identical characteristics, so the distribution of agents' characteristics does not change over time. This simplifying assumption allows us to focus on the main messages while keeping the exposition simple. As we show in Section 5, when considering a richer model with exogenous inflow in which married agents are not replaced by clones, there indeed exists a steady-state equilibrium in which the distribution of agents' characteristics does not change over time.

A stationary strategy for agent a, $A(a) \subseteq [0,1]$, is simply the set of acceptable partners for a, which we assume to be measurable. A strategy profile $\mathcal{A}$ is a collection of individual strategies. For each agent $a \in [0,1]$, a strategy profile yields a unique match set $M(a)$, which is the set of mutually acceptable partners of a, formally $M(A) = A(a) \cap \{b : a \in A(b)\}$.

Let V_o be the corresponding value of search. To calculate the search value, we need to specify the contact probabilities. We employ a quadratic matching technology with a Poisson process with the flow rate μ_o to specify the contact probabilities. We assume that the contact flow probabilities are independently and identically distributed over all individuals. Then, for any strategy profile $\mathcal{A}$ where the induced matching set is $M(a)$, we have

$$V_o(a;\mathcal{A}) = e^{-r\Delta}\left(e^{-\mu_o \int_{M(a)} dF(x)\Delta}\right) V_o(a;\mathcal{A})$$
$$+e^{-r\Delta}\mu_o \int_{M(a)} dF(x)\Delta \left(e^{-\mu_o \int_{M(a)} dF(x)\Delta}\right) \frac{\int_{M(a)} x dF(x)}{\int_{M(a)} dF(x)} + \mathcal{O}(\Delta),$$

where $\mathcal{O}(\Delta)$ is the collection of all terms with $\lim_{\Delta\to 0} \frac{\mathcal{O}(\Delta)}{\Delta} = 0$. Existence and uniqueness of $V_o(a;\mathcal{A})$ follows from the standard arguments using the Contraction Mapping Theorem.

Rearranging the terms we get

$$V_o(a;\mathcal{A})\frac{1-e^{-\left(r+\mu_o\int_{M(a)}dF(x)\right)\Delta}}{\Delta}=\left(e^{-\mu_o\int_{M(a)}dF(x)\Delta}\right)\mu_o\int_{M(a)}xdF(x)+\frac{\mathcal{O}(\Delta)}{\Delta},$$

which yields the following expression as $\Delta\to 0$

$$V_o(a;\mathcal{A})=\frac{\mu_o\int_{M(a)}bdF(b)}{r+\mu_o\int_{M(a)}dF(b)} \tag{1}$$

Definition 1. *A search equilibrium is a strategy profile $\mathcal{A}$ with the induced matching set $M(\cdot)$ where for any a, $b\in A(a)$ if and only if the value of search for a is not more than b; that is, $V_o(a:\mathcal{A})\leq b$.*

Proposition 1. *There exists a unique search equilibrium strategy profile $\mathcal{A}$. The equilibrium can be described with a decreasing sequence of thresholds $\{k_n\}_{n\geq 0}$ with $k_0=1$ where for any $n\geq 1$ and for any $a\in[k_{n-1},k_n]$,*

$$V_o(a,\mathcal{A})=\frac{\mu_o\int_{k_n}^{k_{n-1}}bdF(b)}{r+\mu_o\int_{k_n}^{k_{n-1}}dF(b)}=k_n.$$

This is a well-known result. Its different versions and generalizations are proved in McNamara and Collins (1990), Burdett and Coles (1997), Bloch and Ryder (2000), Smith (2006), Antler and Bachi (2022). The proof proceeds recursively starting with the acceptable partner decision of the top type by showing that there is a connected interval of agents including the top type which uses the same cutoff acceptance strategy where all these agents accept only the same interval of types on the other side. Once the top interval of agents is fixed, the same argument shows a similar result for a second top interval in the remaining population without the first top interval.

3 The platform

There is a matching platform which can potentially facilitate the search process for the agents who are willing to pay a subscription fee for the period they effectively search for a partner using the search channel that the platform provides. Naturally, however, the platform cannot prevent people from contacting and matching with others in their daily life outside the platform, and hence, members can continue to search for a partner outside through the standard search channel discussed in Section 2.

Each agent can either subscribe to the platform or not. If a does not subscribe, s/he can contact with agents of the opposite sex with pizzazz value in some measurable set M at a flow rate $\mu_o \int_M dF(x)$ exactly as before. If a subscribes, there are two independent ways that a can can contact with agent from the opposite sex; through the platform to all members of the platform and through the standard flow process outside the platform to all potential partners in the society. We assume that these channels are independent of each other. Let $S \subseteq [0,1]$ be any measurable set of subscriber types in the population. Then, an individual of type a can contact with agents of the opposite sex with pizzazz value in some measurable set M at a flow rate $\mu_o \int_M dF(x) + \mu_s \int_{M\cap S} dF(x)$. This additive flow rate follows from the independence of the contacts from Platform and from outside (See Ross, 1995). Therefore, the subscription decisions of a measurable set of agents do not only increase their search value by increasing the probability of contact for them but it also increases the value of the subscribers who deem them acceptable, giving them an incentive to subscribe as well.

The platform charges a flow price ρ which amounts to the total cost for a subscriber if s/he becomes a member for a unit interval. Consequently, agents evaluate the discounted cost of becoming a member for the time interval $[0,\Delta]$ as $-\rho \int_0^\Delta e^{-rt}dt$. An agent a's stationary strategy is a tuple $\sigma_a = (\tau_a, A(a))$ consisting of i) a binary subscription decision $\tau_a \in \{0,1\}$ where 1 represents subscribing and 0 represents not subscribing, ii) a measurable set of acceptable partners $A(a)$. A strategy profile σ induces (measurable) set of matching set $M^\sigma(a)$ for each a, a set of subscribers S^σ which we assume to be measurable, and the related set of non-subscribers $(S^\sigma)^c = [0,1]\backslash S$. As long as there is no ambiguity about the strategy profile, we simply write $M(a)$, S, and S^c for the induced matching sets, and the sets of subscribers and non-subscribers.

With this framework, the search values when there is a platform can be calculated as given by the Lemma 1 below.

Lemma 1. *Fix any strategy profile σ. For any $a \in S^c$ (a non-subscriber type at this strategy profile) and for any $a' \in S$ (a subscriber at this profile), the value of search for a and a' are respectively as follows:*

$$V_{ns}(a;\sigma) = \frac{\mu_o \int_{M(a)} x dF(x)}{r + \mu_o \int_{M(a)} dF(x)} \tag{2}$$

$$V_s(a', \rho; \sigma) = \frac{-\rho + \mu_s \int_{M(a') \cap S} x dF(x) + \mu_o \int_{M(a')} x dF(x)}{r + \mu_s \int_{M(a') \cap S} dF(x) + \mu_o \int_{M(a')} dF(x)} \tag{3}$$

Given that the search value for each type can be unambiguously defined for each strategy profile, a participation equilibrium concerning stationary strategies can be defined merely by the optimality condition for each type's decision.

Definition 2. *Fix $\mu_s > 0$ and $\rho > 0$. A* ***participation equilibrium*** *is a strategy profile σ where,*

- *the acceptance set is optimal for each type given their subscription decision:*
  - *for any $a \in S$, $b \in A(a)$ if and only if $V_s(a, \rho; \sigma) \leq b$,*
  - *for any $a \notin S$, $b \in A(a)$ if and only if $V_{ns}(a; \sigma) \leq b$;*
- *participation decision for each type is the optimal one:*
  - *for any $a \in S$, $V_s(a, \rho, \sigma) \geq V_{ns}(a; (\sigma'_a, \sigma_a))$ for any $\sigma'_a = (0, \cdot)$*
  - *for any $a \notin S$ $V_{ns}(a, \rho) \geq V_s(a; (\sigma'_a, \sigma_a))$ for any $\sigma'_a = (1, \cdot)$*

Note that in any participation equilibrium, the set of subscribers is also determined endogenously. The expectations of individuals about set of subscribers should in turn be consistent with the actual set of subscribers realized in any participation equilibrium. Inspecting the search value of subscriber as described in equation (3) reveals that there always exists a participation equilibrium with no subscription; that is, with $S = \emptyset$. When the price is too high, no subscription is the only equilibrium. But, when price is lower there are also equilibria with maximal equilibrium. To concentrate on the equilibria with the highest possible subscription, we define below "sequentially maximal participation equilibrium." The aim is to identify equilibria where there are no coordination failures so that for instance if the top cluster benefits from subscription as a whole they should be subscribing. By the theorem below, we first describe the set of all equilibria.

Theorem 1. *For all $\mu_s > 0$ and $\rho > 0$, there exist a participation equilibrium. In any participation equilibrium, there exist a decreasing sequence $\{\hat{k}_i\}_{i \geq 0}$ with $\hat{k}_0 = 1$ such that for all $i \geq 1$ and for all $0 < \hat{k}_i < \hat{k}_{i-1}$, either $[\hat{k}_i, \hat{k}_{i-1}] \subseteq S$ or $[\hat{k}_i, \hat{k}_{i-1}] \cap S = \emptyset$. Furthermore, for all $\mu_s > 0$, there is ρ such that there is a participation equilibrium with $S \neq \emptyset$.*

Proof As each agent has the same ordinal ranking of partners with type 1 being the best, for any type a and for any acceptance strategy $A(a)$, if $b \in A(a), b < b'$ and $b' \notin A(a)$, including b' to the acceptance set would never hurt type a. Therefore, for any participation equilibrium in undominated strategies and for all $a \in [0,1]$, $A_s(a) = [k_a, 1]$ or $A_{ns}(a) = [o_a, 1]$ where k_a is the lowest type a accepts when s/he subscribes and o_a is the lowest type a accepts when s/he does not subscribe and searches only outside. Therefore, an equilibrium strategy for type a can be characterized by the subscription decision and the lowest acceptance threshold type, i.e., either $\sigma_a = (1, k_a)$ or $\sigma_a = (0, o_a)$.

Let σ be a participation equilibrium in undominated strategies and S be the induced set of subscribers. Then the search value for 1 in case s/he does not subscribe and uses a threshold $o \in [0,1]$ does not depend on S (or σ_{-1}) as 1 is acceptable for all types regardless of their subscription decision (and hence, $M(1) = [o,1]$), and it can be calculated as follows.

$$V_{ns}(1;o) = \frac{\mu_o \int_{[o,1]} x dF(x)}{r + \mu_o \int_{[o,1]} dF(x)}.$$

The following lemma is straightforward to prove:

Lemma 2. *The equation $V_{ns}(1;o) = o$ has a unique solution $o_1 \in (0,1)$.*

On the other hand, the search value for 1 when he subscribes depends on σ_{-1} only through S. We first show that there is a unique optimal threshold for 1 given S. Now, let 1 subscribe and employ the threshold $k \in (0,1)$. The search value denoted by $V_s(1,k|S)$ is as follows.

$$V_s(1,k|S) = \frac{-\rho + \mu_s \int_{[k,1]\cap S} x dF(x) + \mu_o \int_{[k,1]\setminus S} x dF(x)}{r + +\mu_s \int_{[k,1]\cap S} dF(x) + \mu_o \int_{[k,1]\setminus S} dF(x)}.$$

Lemma 3 below implies that if there exists a solution to the indifference condition $V_s(1,k|S) = k$, it is unique.

Lemma 3. *Suppose that $k_1 \in (0,1)$ is such that $V_s(1,k_1|S) = k_1$, then $\frac{\partial V_s(1,k_1|S)}{\partial k} = 0$. Therefore, k_1 is unique.*

Proof The derivative $\frac{\partial V_s(1,k_1|S)}{\partial k}$ can be calculated as

$$\frac{\partial V_s(1,k_1|S)}{\partial k} = \frac{-\hat{\mu} k f(k) \hat{D} + \hat{\mu} f(k) \hat{N}}{\hat{D}^2},$$

where $\hat{\mu} = \mu_s$ if $k \in S$ and $\hat{\mu} = \mu_o$ if $k \notin S$; $\hat{N}$ is nominator of $V_s(1, k_1|S)$ and $\hat{D}$ is the denominator. As $f(k) > 0$ for any $k \in [0, 1]$, when $k = V_s(1, k_1|S) = \frac{\hat{N}}{\hat{D}}$, we have $\frac{\partial V_s(1,k_1|S)}{\partial k} = 0$. □

Now we show that all the agents in 1's match set copy type 1's strategy.

Lemma 4. *If $\sigma_1 = (0, o_1)$, then for all $a \in [o_1, 1]$ we have $\sigma_a = (0, o_1)$. If $\sigma_1 = (1, k_1)$, then for all $a \in [k_1, 1]$ we have $\sigma_a = (1, k_1)$.*

Proof First, note that if some $a \in (0, 1]$ accepts a type $b \in [0, 1]$ as a partner, so does any $a' < a$ whose subscription decision is the same as a. We first show that if type 1 does not subscribe, no type in $[o_1, 1]$ subscribes and if type 1 subscribes, all types in $[k_1, 1]$ subscribes.

Let $\sigma_1 = (0, o_1)$ and suppose for a contradiction that there is $a \in [o_1, 1] \cap S$. Then, all types in $[o_1, 1]$ are also acceptable for all non-subscribers. Since we assume that agents subscribe in case of an indifference, a maximum type in S exists, say $\hat{a}$. As otherwise, in case $\hat{a} \in S^c$ is the supremum but not the maximum, $\hat{a}$ would deviate and increase her/his search value, contradicting that σ is an equilibrium. Note that $\hat{a}$ is acceptable by all types in the platform as otherwise paying the subscription price is not rational. Therefore, $\hat{a}$ is acceptable by all types. If $V_{ns}(1, o_1) \leq V_s(\hat{a}, \sigma)$, type 1 would deviate by copying $\hat{a}$'s strategy. If $V_s(\hat{a}, \sigma) < V_{ns}(1, o_1)$, then type $\hat{a}$ would deviate by copying 1's strategy and receiving $V_{ns}(\hat{a}, o_1) = V_{ns}(1, o_1)$ as $\hat{a}$ is acceptable by all types.

Now, let $\sigma_1 = (1, k_1)$ and suppose for a contradiction that there is $a \in [k_1, 1] \cap S^c$. Then, all types in $[k_1, 1]$ are also acceptable for all subscribers. Let $\hat{a}$ be the supremum non-subscriber type. Then, there is b sufficiently close enough to $\hat{a}$ such that b is acceptable by all non-subscribers. Therefore, b is acceptable by all types. If $V_s(1, k_1) < V_{ns}(b, \sigma)$, type 1 would deviate by copying b's strategy. If $V_{ns}(\hat{a}, \sigma) \leq V_s(1, k_1)$, then type b would deviate by copying 1's strategy and receiving $V_{ns}(\hat{a}, o_1) = V_{ns}(1, o_1)$ as $\hat{a}$ is acceptable by all types.

If $\sigma_1 = (0, o_1)$, $a \in [o_1, 1]$ is acceptable for all types. Therefore, a can attain the maximum possible value equal to $V_{ns} = (1; o_1)$ by setting $\sigma_a = (0, o_1)$. For a similar reason, $\sigma_a = (0, k_1)$ for all $a \in [k_1, 1]$ if 1 subscribes. □

Considering only the subscription option, if the price ρ is large enough, the search value of 1 will never be large enough to reject a contact. However, there is small enough price for which the threshold type that defines 1's acceptance set always exist and it is unique by Lemma 3.

Lemma 5. *If $\rho \geq (\mu_s + \mu_o)\phi$, $V_s(1, k|S) < k$ for any $k \in [0, 1]$. If $\rho < (\mu_s + \mu_o)\phi$, there is a unique $k_1 \in (0, 1)$ with $k_1 = V_s(1, k|S)$.*

Proof As the price is too high, we know that $V_s(1, 0|S) < 0$ and $V_s(1, 1|S) < 0$. Now suppose to the contrary that there is $\underline{k} \in (0, 1)$ such that $V_s(1, \underline{k}|S) = \underline{k}$, and let $\underline{k}$ be the minimum of such solutions. Then, for any $\varepsilon > 0$ small enough

$$\frac{\partial V_s(1, \underline{k}|S)}{\partial k} = \lim_{\varepsilon \to 0} \frac{V_s(1, \underline{k}|S) - V_s(1, \underline{k} - \varepsilon|S)}{\varepsilon} > \frac{\underline{k} - (\underline{k} - \varepsilon)}{\epsilon} = 1.$$

Nevertheless, this contradicts with Lemma 3.

If $\rho < (\mu_s + \mu_o)\phi$, $V_s(1, 0|S) > 0$, there is a solution and by Lemma 3 the solution is unique. □

By Lemma 4, there are only two cases in any participation equilibrium: either $[k_1, 1] \subseteq S$ or $[o_1, 1] \cap S = \emptyset$. Furthermore, the participation constraint implies that when $[k_1, 1] \subseteq S$, $k_1 \geq o_1$. Define $\hat{k}_1 = k_1$ if 1 subscribes and $\hat{k}_1 = o_1$ if 1 does not subscribe.

Now as the inductive hypothesis, for any $i > 1$ fix any $\hat{k}_{i-1} \in (0, 1)$ such that $\bigcup_{a \in [\hat{k}_{i-1}, 1]} M(a) = [\hat{k}_{i-1}, 1]$. Now, we can replicate the arguments above for the top cluster in the remaining population of $[0, k_{i-1}]$ to show that there exist uniquely defined k_i and o_i and either of the following is true $[k_i, k_{i-1}] \subseteq S$ or $[o_i, k_{i-1}] \cap S = \emptyset$. ■

A result of Theorem 1 is that the subscribers on equilibrium only accept among each other even if they have the option of searching outside the platform.

Corollary 1. *Fix $\mu_s > 0$ and $\rho > 0$. In any participation equilibrium with a set of subscribers S, $a \in S$ implies $M(a) \subseteq S$.*

Figure 1 illustrates the set of participation equilibria at an interior price $\rho = 0.5$. The top clusters can subscribe in a participation equilibrium. But there are two more participation equilibria. In one of them, only the top cluster subscribes, while in the last one there are no subscribers. In principle, there could have been a participation equilibrium where only the second cluster subscribes. However, as demonstrated on Figure 1, the search value for the second cluster, given by $k_2(o_1) = 0.185$, is lower than the value of searching outside $o_2 = 0.191$. Therefore, there is no participation equilibrium where the top cluster does not subscribe but the second one does.

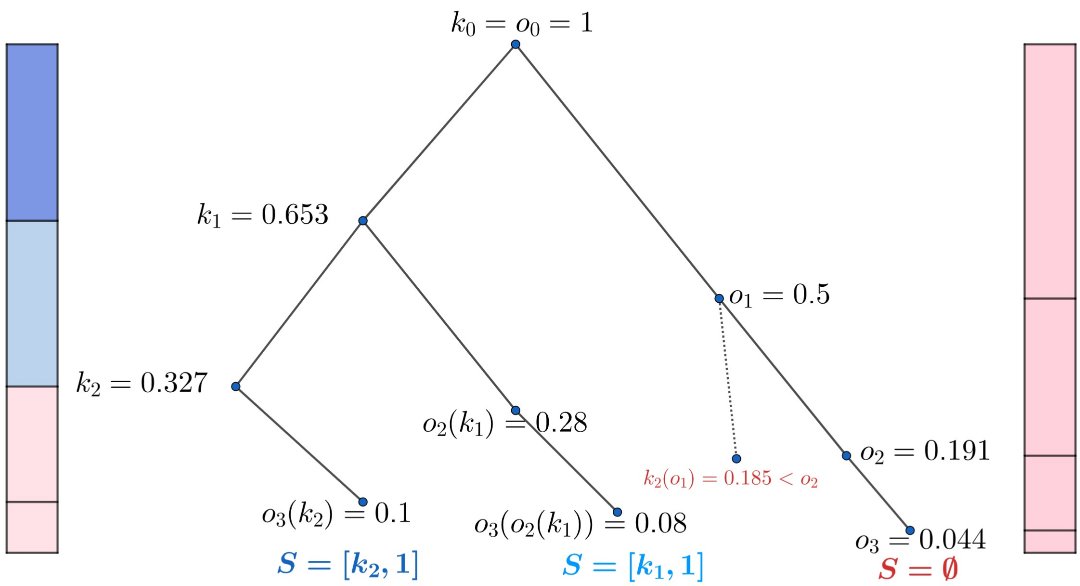

Figure 1: There are three participation equilibria when $F(x) = x$, $\mu_o = 1$, $\mu_s = 10$, $r = 0.25$, and $\rho = 0.5$.

4 Demand Function

To understand the pricing of the monopolist, we need to construct a demand function. However, since for each price $\rho > 0$ there could be multiple participation equilibria, we need to fix the equilibrium expectations of the monopolist. We concentrate on the type of equilibria without coordination failures. Definition 3 formalizes the type of equilibria we concentrate.

Definition 3. *Fix $\mu_I > 0$ and $\rho > 0$. A sequentially maximal participation equilibrium is a participation equilibrium with a corresponding subscription set S that satisfies the following: for any $\bar{k} \in (0, 1)$ such that for any $a < \bar{k}$ $M(a) \subseteq [0, \bar{k}]$ and $a > \bar{k}$ $M(a) \subseteq [\bar{k}, 1]$ and a corresponding top cluster $[\hat{k}_1(\bar{k}), \bar{k}]$, there is no other participation equilibrium with a corresponding subscription set S' such that $(S \cap [\hat{k}_1(\bar{k}), \bar{k}]) \subsetneq (S' \cap [\hat{k}_1(\bar{k}), \bar{k}])$.*

In a sequentially maximal equilibrium, we first check if the top cluster benefits from subscription in case all of them subscribe. If they do, we include the top cluster that corresponds to their full subscription to the subscriber set. Given the behavior of the top cluster, we move to the second cluster and so on and so forth. This recursive construction leads to the following result.

Theorem 2. *Fix any $\mu_I > 0$ and $\rho > 0$. There exists a unique sequentially maximal participation equilibrium. This equilibrium can be described by a finite sequence $\{\hat{k}_i\}_{i=0}^n$ for some positive integer n, where $\hat{k}_0 = 1$ and $\hat{k}_i = \max\{k_i, o_i\}$. For each $i \geq 1$ the thresholds k_i and o_i satisfy*

$$V_s^i = \frac{-\rho + (\mu_s + \mu_o)\int_{k_i}^{\hat{k}_{i-1}} x dF(x)}{r + (\mu_s + \mu_o)\int_{k_i}^{\hat{k}_{i-1}} dF(x)} = k_i \quad \& \quad V_{ns}^i = \frac{\mu_o \int_{o_i}^{\hat{k}_{i-1}} dF(x)}{r + \mu_o \int_{o_i}^{\hat{k}_{i-1}} x dF(x)} = o_i. \tag{4}$$

The set of subscribers is defined as follows: the interval $[k_i, \hat{k}_{i-1}] \subseteq S$ if and only if $k_i \geq o_i$.

Proof We can again start with the very top members of the society. When the expectations are favorable, the top cluster should subscribe if it leads to more payoff to do so. the following Lemma proves that the value of subscription increases with the size of the expected subscription set.

Lemma 6. *Fix any $\mu_I, \rho > 0$ and the matching function $M(\cdot)$. Let S and $\hat{S}$ be two measurable subsets of $[0, 1]$ and let $V_s(\cdot, S)$ and $V_s(\cdot, \hat{S})$ be corresponding search functions defined as in equation (3). Then $S \subseteq \hat{S}$ implies $V_s(\cdot, S) \leq V_s(\cdot, \hat{S})$.*

Proof For any $a \in [0, 1]$ let $V_s(a, S) = \frac{N(a,S)}{D(a,S)}$ so that

$$N(a, S) = -\rho + \mu_I \int_{M(a)\cap S} b dF(b) + \mu_o \int_{M(a)} b dF(b),$$
$$D(a, S) = r + \mu_I \int_{M(a)\cap S} dF(b) + \mu_o \int_{M(a)} dF(b).$$

The terms $N(a, \hat{S})$ and $D(a, \hat{S})$ are defined similarly. Then,

$$V_s(a, \hat{S}) = \frac{N(a, S) + \mu_I \int_{M(a)\cap(\hat{S}\setminus S)} b dF(b)}{D(a, S) + \mu_I \int_{M(a)\cap(\hat{S}\setminus S)} dF(b)}$$
$$\geq \frac{N(a, S) + V_s(a, \hat{S})\mu_I \int_{M(a)\cap(\hat{S}\setminus S)} dF(b)}{D(a, S) + \mu_I \int_{M(a)\cap(\hat{S}\setminus S)} dF(b)} = V_s(a, \hat{S}),$$

where the inequality follows from $V_s(a, \hat{S}) \leq b$ for any $b \in M(a)$. □

With Lemma 6, we know that for any acceptance set by the universally acceptable top members as the expected set of subscribers increase in set inclusion, the search value of the top members increase. If the top cluster expects all (or enough) members to subscribe, the corresponding subscription acceptance set would be defined by a k_1 that satisfies equation (4). Note that by Lemmas 2 and 5, there exists unique thresholds o_1 and k_1 that satisfy the equation (4).

If $k_1 \geq o_1$, the participation search value is also higher than the nonparticipation search value. Then a sequentially maximal participation equilibrium would correspond to $[k_1, 1] \subseteq S$. If $k_1 < o_1$, the top cluster members do not want to subscribe even if they expect enough people to subscribe. By Lemma 6 not subscription becomes a dominant strategy for the members of $[o_1, 1]$.

Now, given the behavior of the top cluster, we can repeat the same arguments to show that the second cluster subscribes as a whole if $k_2(\hat{k}_1) \geq o_2(\hat{k}_1)$, while a straightforward generalization of Lemmas 2 and 5 show the uniqueness and existence of these thresholds. Continuing recursively proves the first part of the statement.

Next, the following Lemma is required to prove the the process ends in finite steps.

Lemma 7. *Fix any $\mu_I > 0$ and $\bar{k} \in (0,1]$ so that $\bigcup_{a \in [\bar{k},1]} M(a) = [\bar{k}, 1]$. If $\rho > (\mu_I + \mu_o) \int_0^{\bar{k}} b dF(b)$, there is no subscription from $[0, \bar{k}]$ in any participation equilibrium.*

Proof As $\bigcup_{a \in [\bar{k},1]} M(a) = [\bar{k}, 1]$, for any type $a < \bar{k}$ close enough to $\bar{k}$ should be accepted by all types in $[0, \bar{k}]$. By Lemma 6, for every acceptance set chosen by the top cluster in $[0, \bar{k}]$, if the expected set of subscribers expand, the resulting search value of subscription increases. This also implies by an optimality argument that the search value of subscription at the optimal choice of the acceptance set also increases as the expected set of subscription expands. As the search values of all types in the same cluster are the same, then at each expected subscription set either all of the top cluster type subscribe or none of them subscribes. Then, with the expectation of all of them subscribing; the search value given the remaining population $[0, \bar{k}]$ is given as

$$V_s^1(k; \bar{k}) = \frac{-\rho + (\mu_I + \mu_o) \int_k^{\bar{k}} b dF(b)}{r + (\mu_I + \mu_o) \int_k^{\bar{k}} dF(b)}.$$

Since the price is too high, we know that $V_s^1(0; \bar{k}) < 0$ and $V_s^1(\bar{k}; \bar{k}) < 0$. Suppose to the contrary that there is $\underline{k} = V_s^1(\underline{k}; \bar{k})$, and let $\underline{k}$ be the minimum of such solutions. Then, for any $\varepsilon > 0$ small enough

$$\frac{\partial V_s^1(\underline{k}; \bar{k})}{\partial \underline{k}} = \lim_{\epsilon \to 0} \frac{V_s^1(\underline{k}; \bar{k}) - V_s^1(\underline{k} - \epsilon; \bar{k})}{\epsilon} > \frac{\underline{k} - (k - \varepsilon)}{\varepsilon} = 1.$$

But this contradicts with $\frac{\partial V_s^1(\underline{k};\bar{k})}{\partial \underline{k}} = 0$ at any solution $\underline{k}$. □

To prove for any $\rho > 0$ this process ends in finite steps, first note that $\hat{k}_i$ converges to 0 as $i \to 0$ and for any $\rho > 0$ there is $\bar{k} \in [0, 1]$ such that $\rho > (\mu_I + \mu_o) \int_0^{\bar{k}} b dF(b)$ and so there is no subscription among $[0, \bar{k}]$ by the lemma above. ■

Theorem 2 provides an algorithm to construct the demand function.

Definition 4. *Fix $\mu_I > 0$. For any $\rho > 0$ let S correspond to the set of subscribers in the unique sequentially maximal participation equilibrium. The demand function for a fixed mu_s is defined as $D(\rho, \mu_s) = 2 \int_S dF(x)$; twice of the measure of the set of subscribers.*

Theorem 2 shows that the demand function can be written as

$$D(\rho, \mu_s) = \sum_{i \geq 1 : k_i \geq o_i} \int_{k_i}^{\hat{k}_{i-1}} dF(x)$$

The optimal pricing by the platform at each instant should maximize the revenue flow to the platform. For any μ_s and $\rho > 0$ the revenue flow to the platform can be written as $D(\rho)\rho$. Let $\rho(\mu_s)$ be any revenue flow maximizing stationary strategy by the platform. Assuming a common discount factor $r > 0$, the long-term profit of the platform from a stationary sequentially maximal participation equilibrium would be $V_{pi}(\rho(\mu_s), \mu_s) = \frac{D(\rho(\mu_s))\rho(\mu_s)}{r}$.

If the demand function $D(\rho)$ was differentiable, we would be able calculate the revenue flow maximizing price from possibly from the first-order conditions to find an interior optimal price. However, as the matching sets assume a class-structure, demand function may not be differentiable even if it is continuous. Inspecting the equilibrium conditions for the thresholds $\{k_i\}_{i \geq 1}$ and $\{o_i\}_{i \geq 1}$, given a sequentially maximal participation equilibrium, we can recognize that a price can be called as an "interior" price for some cluster $i \geq 1$ if $k_i \neq o_i$ and as a "corner" price if $k_i = o_i$. We say a price $\rho > 0$ is "completely interior" if at the corresponding sequentially maximal participation equilibrium there is no i such that $k_i = o_i$. Almost all prices are interior prices for some thresholds but only a countably infinite of them are corner prices for some thresholds.

For the discussion below considering a particular sequence of corner prices is useful. For any $i \geq 1$ consider a participation equilibrium where none of the clusters except the $i^t h$ one subscribes. And for the $i^t h$ consider the price $\bar{\rho}_i$ such that $k_i(o_{i-1}) = o_i$. Clearly, by construction $\bar{\rho}_i > 0$. Now calculate all such prices $\{\bar{\rho}_i\}_{i \geq 1}$. Finally, let

$\bar{\rho} = \max_i \bar{\rho}_i$. We assume for the discussion below that there is a unique $i \geq 1$ such that $\bar{\rho}_i = \bar{\rho}$.

The first observation we make is the demand function does not always satisfy the "law of demand."

Proposition 2. *Fix $\mu_I > 0$. There is a range of prices $(\underline{\rho}, \bar{\rho})$ such that the demand function is strictly increasing in price in this region. Moreover, the optimal stationary price is never completely interior.*

Proof Fix any $\mu_o > 0$ and the type distribution F. We can uniquely calculate the sequence of thresholds $\{o_i\}_{i \geq 1}$ by solving $V_{ns^i} = o_i$ recursively using equation (4) for matching outside of the platform. Let $\bar{\rho}$ be as defined above. Then, for $\varepsilon > 0$ small enough at the price range $(\bar{p} - \varepsilon, \bar{p})$, the demand function is equal to $o_{i-1} - k_i(\rho)$. The corresponding value of search is V_s^i as defined in equation (4). And, by implicit function theorem

$$\frac{dk^i}{d\rho} = \frac{\frac{\partial V_s^i}{\partial \rho}}{1 - \frac{\partial V_s^i}{\partial k_i}} < 0,$$

as $\frac{\partial V_s^i}{\partial k_i} = 0$ when $k_i = V_s^i$. As for the price range we consider, the type of subscriber clusters do not change, an increase in ρ increases $o_{i-1} - k_i$ and so the demand function. Using the same argument we can make another observation. The optimal price is never a completely interior. ■

Proposition 2 shows that there are regions of prices that the platform can increase its demand by increasing its price. The main difficulty in the proof is to find a range of prices where the type of clusters that collectively subscribe do not change. The existence of such a price range holds because of the pattern of block segregation that we find any participation equilibrium. For any such price where the type of clusters that subscribe do not change, the flow price shows its "interior" effect. As the flow price increases, everyone who subscribes has a higher waiting cost. This makes them more "tolerant" for acceptance; that is, their acceptance sets expands. This motivates the top members from the next outsider cluster to subscribe as well because their subscription search value increases. As a result the equilibrium cluster that subscribes expands, which results in an increase in demand.

Figure 2 illustrates the effect of consumption externalities on the demand function for the case of uniform distribution of types. When the price is high enough, it is not beneficial for any cluster of agents to subscribe. At price $\bar{\rho}_1$, the value of subscription

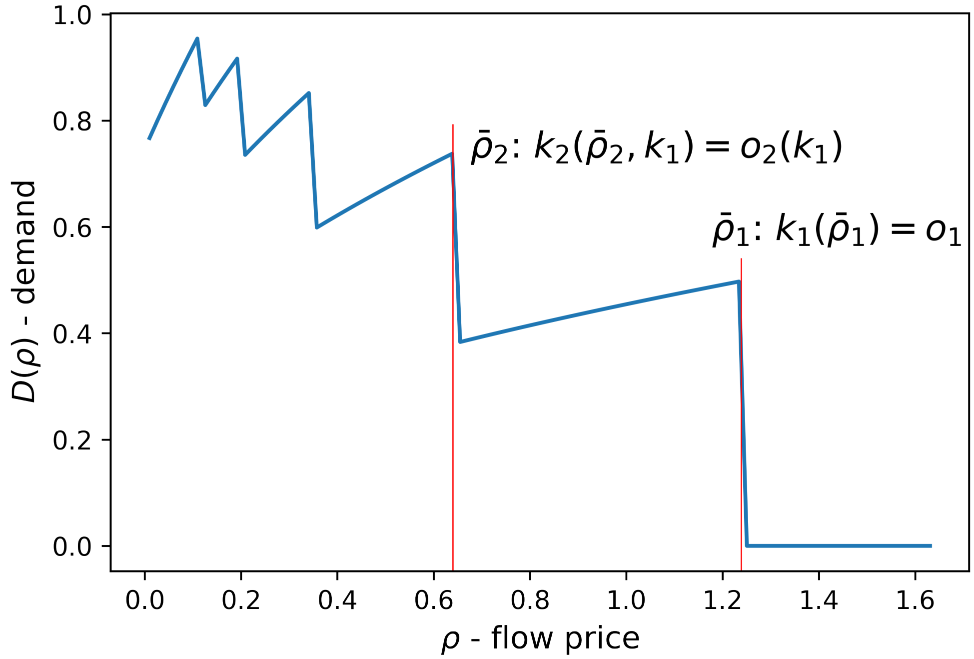

Figure 2: The demand function $D(\rho)$ when $F(x) = x$, $\mu_o = 1$, $\mu_s = 10$, $r = 0.25$

for the top cluster is the same as not subscribing as a whole. At the corresponding sequentially maximal participation equilibrium, the top cluster of agents participate. The price $\bar{\rho}_1$ is too high for the other clusters to benefit from subscription. At the lower price $\bar{\rho}_2$, now the second cluster is indifferent between subscribing or not. When price increases slightly from $\bar{\rho}_2$, the second cluster drops from the set of subscribers. However, at the same time the demand from the top cluster continues to increase as price increases.

Inspecting the demand function illustrated by Figure 2 suggests that the "law of demand" still works at the global scale. If the price increases substantially, the demand is more likely to be lower as some of the clusters drop from the set of subscribers. However, incremental increases in the price could lead to an increase in the demand. Therefore, the "law of demand" does not necessarily hold locally.

The optimal pricing by the platform cannot be at one of the prices where the demand is increasing; but it has to correspond to one of the "peaks" from the demand function. Therefore, the problem of optimal pricing reduces to a discrete optimization problem where the platforms picks the optimal peak point.

Vanishing search frictions:

In a typical decentralized search in a marriage market the search equilibrium matching outcome converges to the stable matching as the search frictions vanish (Bloch and Ryder, 2000; Adachi, 2003). In our environment with a platform, we can ask a related question about the outcome among the subscribers as the search frictions implied by the platform's matching technology vanishes, i.e. as μ_s grows indefinitely. As proved in Theorem 1, a cluster of agents who accept each other as partners either subscribe to the platform as a whole or not at all. So we can concentrate on the matching among the subscriber clusters as μ_s increases. However, the pricing strategy of the platform also determines the equilibrium outcome and in principle the optimal price choice of the platform might depend on its flow rate μ_s.

At the price $\bar{\rho}$ there is a single cluster $[o_i, o_{i-1}]$ for some $i \geq 1$ that subscribes to the platform. Does the matching outcome in $[o_i, o_{i-1}]$ converges to the stable matching as $\mu_s \to \infty$. The answer is negative as the price $\bar{\rho}$ can also increase proportionally with μ_s to keep incentivizing the cluster $[o_i, o_{i-1}]$ to subscribe. Therefore, the matching outcome in all of the society stays constant as $\mu_s \to 0$. Based on this argument, we can prove the following proposition.

Proposition 3. *There is a pricing strategy $\rho(\mu_s)$ that keeps the matching and subscription set constant as μ_s increases. Furthermore, with this strategy the long-term profit of the platform $V_\pi \to \infty$ as $\mu_s \to \infty$.*

Proof Consider the price $\bar{\rho}$ as defined above at which $D(\bar{\rho}) = o_{i-1} - o_i$ for some i. Then, $\bar{\rho}$ can be written by using the equation (4) as follows

$$\bar{\rho} = (\mu_s + \mu_o) \int_{o_i}^{o_{i-1}} x dF(x) - o_i[r + (\mu_s + \mu_o) \int_{o_i}^{o_{i-1}} dF(x)] > 0$$

by construction. As $\{o_i\}_{i \geq 1}$ does not depend on ρ or μ_s, as $\mu_s \to \infty$ the demand stays constant, and $\bar{\rho} \to \infty$. As the revenue maximizing pricing should yield a profit no less than the one with $\bar{\rho}$, the profit at the optimal pricing strategy also increases to ∞. ■

At the price $\bar{\rho}$, the matching set does not change as μ_s increases, as $\bar{\rho}$ also increases proportionally. At very high flow rates $\mu_s > 0$, the members of the platform choose to subscribe because they can expect to match with an acceptable partner in a very short amount of time. As the expected waiting time is very short, they are willing to pay a very high price to the platform. The reason that the platform's profit diverges is due to the assumption that the matched types are replaced immediately with their exact replicas. The platform's revenue from a single batch of types does not necessarily increase because each single batch waits less. However, the platform matches more batches in a given time as μ_s increases and is able to charge more for subscription. To keep up the platform's rapid matching, the inflow rate endogenously adjusts. This exercise shows that the seemingly technical assumption of "endogenous flow" as it is often treated as such in the literature, leads to a "paradox" of infinite profit. In the next section, we consider the model with exogenous flow to address this issue.

5 Exogenous Inflow

Following Smith (2006) we now consider a model with exogenous flow of types and dissolution of matches. Let $G(\cdot)$ be the exogenous c.d.f of types, and $g(\cdot)$ is the corresponding p.d.f. In any stationary strategy profile of the agents and the platform, there is an endogenous distribution of unmatched agents. Let $F(\cdot)$ and $f(\cdot)$ be the c.d.f. and p.d.f. of the unmatched types.

5.1 Benchmark: No Platform

Let $\mu_o > 0$ be the flow of unmatched agents and let $\delta > 0$ be the flow of the dissolution opportunities. Then, having a stationary distribution of unmatched requires the following balance condition to be satisfied at each type $a \in (0, 1)$

$$\delta(g(a) - f(a)) = \mu_o f(a) \int_{M(a)} f(x)dx, \tag{5}$$

where $M(a)$ is the opportunity set of the type a.

The balance condition above puts an upper-limit to the expression $\mu_o f(a)$.

Lemma 8. *As $\mu_o \to \infty$, $f(a) \to 0$ for any $a \in (0, 1)$.*

Proof Left-hand side of equation (5) is bounded by $\delta g(a)$, which is finite. Therefore, as $\mu_o \to \infty$, $f(a)$ cannot converge to anything positive. ■

The value of matching with any type b is b if there was no dissolution probability. Then the corresponding flow payoff can be written as rx. However, with matching dissolution it should be calculated by using Bellman equations. For any type $a \in (0, 1)$, let $V_s(a)$ be the value of search and let $V_m(a, b)$ be the value of matching with the type

x.

$$V_m(a,b) = rb\int_0^{\Delta} e^{-rt}dt + e^{-r\Delta}\left(e^{-\delta\Delta}V_m(a,b) + (1-e^{-\delta\Delta})V_s(a)\right)$$
$$V_m(a,b)\left(\frac{1-e^{-(\delta+r)\Delta}}{\Delta}\right) = rb\frac{1-e^{-r\Delta}}{r\Delta} + e^{-r\Delta}\frac{1-e^{-\delta\Delta}}{\Delta}V_s(a).$$

At the limit as $\Delta \to 0$,

$$V_m(a,b)(\delta + r) = rb + \delta V_s(a).$$

Therefore, the match value can be written as

$$V_m(a,b) = \frac{rb + \delta V_s(a)}{r+\delta}. \tag{6}$$

For a fixed flow μ_o, the search value for any type $a \in (0,1)$ and his/her opportunity set $M(a)$

$$V_s(a) = \frac{\mu_0 \int_{M(a)} V_m(a,x)dF(x)}{r + \mu_o \int_{M(a)} dF(x)} \tag{7}$$

Combining the two equations above, we have

$$V_s(a) = \frac{\mu_0 \int_{M(a)} \frac{rx+\delta V_s(a)}{r+\delta}dF(x)}{r + \mu_o \int_{M(a)} dF(x)} = \frac{\frac{\mu_o r \int_{M(a)} xdF(x)}{r+\delta} + \frac{\mu_o \delta V_s(a) \int_{M(a)} dF(x)}{r+\delta}}{r + \mu_o \int_{M(a)} dF(x)}$$
$$V_s(a)\left(1 - \frac{\mu_o \delta \int_{M(a)} dF(x)}{(r+\delta)(r+\mu_o \int_{M(a)} dF(x))}\right) = \frac{\mu_o r \int_{M(a)} xdF(x)}{(r+\delta)(r+\mu_o \int_{M(a)} dF(x))} \Rightarrow$$
$$V_s(a) = \frac{\mu_o r \int_{M(a)} xdF(x)}{r^2 + r\delta + r\mu_o \int_{M(a)} dF(x)},$$

which simplifies into

$$V_s(a) = \frac{\mu_o \int_{M(a)} xdF(x)}{r + \delta + \mu_o \int_{M(a)} dF(x)} \tag{8}$$

Going back to the match value, we have

$$V_m(a,b) = \frac{rb + \delta\frac{\mu_o \int_{M(a)} xdF(x)}{r+\delta+\mu_o \int_{M(a)} xdF(x)}}{r+\delta} = \frac{rb\left(r+\delta+\mu_o \int_{M(a)} xdF(x)\right) + \delta\mu_o \int_{M(a)} xdF(x)}{(r+\delta)\left(r+\delta+\mu_o \int_{M(a)} xdF(x)\right)}$$

A search equilibrium is a strategy profile $\mathcal{A}$ with the induced matching set $M(\cdot)$

where for any a, $b \in A(a)$ if and only if the value of search for a is not more than b; that is, $V_o(a : \mathcal{A}) \leq b$.

Based on these, we can define a search equilibrium as follows:

Definition 5. *A **search equilibrium** is a strategy profile $\mathcal{A}$ with a corresponding induced matching set $M(\cdot)$, search value function $V_o(\cdot, \mathcal{A})$, match value function $V_o(\cdot, \cdot, \mathcal{A})$, and an unmatched type distribution f such that for any $a, b \in [0, 1]$ $V_o(a, \mathcal{A})$ satisfies equation (7), $V_m(a, b, \mathcal{A})$ satisfies equation (12), the unmatched-type distribution $f(a)$ satisfies equation (5), and finally for any $a \in [0, 1]$ and $b \in M(a)$ $V_o(a, \mathcal{A}) \leq V_m(a, b, \mathcal{A})$.*

The existence of a search equilibrium follows from Smith (2006). For completeness, we present a recursive argument below that works in our environment with block-segregation.

Proposition 4. *Fix any $\mu_o, \delta, r > 0$, the type distribution $g(\cdot)$. There exists a search equilibrium. Any search equilibrium can be described by a sequence $\{k_n\}_{n\geq 0}$ with $k_0 = 1$ and a piece-wise continuous unmatched type distribution function $f(\cdot)$ such that for any $n \geq 1$*

$$\frac{\mu_o \int_{k_n}^{k_{n-1}} x dF(x)}{r + \delta + \mu_o \int_{k_n}^{k_{n-1}} dF(x)} = k_n, \quad \textit{and} \tag{9}$$

$$\delta(g(a) - f(a)) = \mu_o f(a) \int_{k_n}^{k_{n-1}} f(x) dx. \tag{10}$$

Proof Let's start with the top cluster. Given $f(\cdot)$ a top member with pizzazz value 1 would accept a partner of type $k \in (0, 1]$ if

$$\frac{rk + \delta V_s(1)}{r + \delta} \geq V_s(1) \Leftrightarrow V_s(1) \leq k,$$

which is exactly the same condition as the no-separation benchmark in the previous section. Therefore, we can uniquely find $k_1 \in (0, 1)$ such that

$$V_s^1 = \frac{\mu_o \int_{k_1}^{1} x dF(x)}{r + \delta + \mu_o \int_{k_1}^{1} dF(x)} = k_1.$$

The corresponding balance equation is for any $a \in [k_1, 1]$

$$\delta(g(a) - f(a)) = \mu_o f(a) \int_{k_1}^{1} dF(x).$$

Now, the existence of the search parameters for the top cluster is no longer straightforward as before because k_1 and $f(\cdot)$ depend on each other. Lemma 9 below proves the existence of a search equilibrium outcome for the top-cluster.

Lemma 9. *Fix any $\mu_o, \delta, r > 0$ and the type distribution $g(\cdot)$. There exists a threshold $k_1 \in (0,1)$ and $f(\cdot) \in C([0,1]|g)$, where $C([0,1]|g)$ is the space of continuous functions on $[0,1]$ that are bounded between 0 and g, such that for any $a \in [0,1]$ $0 \leq f(a) \leq g(a)$ and $k_1, f(\cdot)$ uniquely describe the search equilibrium outcome for the top cluster.*

Proof First, note that for any fixed $0 \leq f(\cdot) \leq g(\cdot)$, the following equation has a unique solution $k_1 \in (0,1)$.

$$\frac{\mu_o \int_{k_1}^{1} x dF(x)}{r + \delta + \mu_o \int_{k_1}^{1} dF(x)} = k_1.$$

Now, consider the balance equation for the types $a \in [k_1, 1]$

$$\delta(g(a) - f(a)) = \mu_o f(a) \int_{k_1}^{1} f(x) dx.$$

Let $\gamma_f \equiv \int_{k_1}^{1} f(x) dx$. Integrating the equation above over $[k_1, 1]$ yields

$$\delta(1 - G(k_1)) = \delta \gamma_f + \mu_o \gamma_f^2,$$

which has a unique solution that satisfies $\gamma_f \in (0, 1 - G(k_1))$.

Now, we can define a mapping $T : C([0,1]|g) \to C([0,1]|g)$ such that for any $f \in C([0,1]|g)$ a corresponding uniquely defined k_1 and γ_f as above let

$$T(f)(a) = \frac{\delta g(a)}{\delta + \mu_o \gamma_f}.$$

Note that the mapping T is well-defined. Next, the space $C([0,1]|g)$ is a convex and compact metric space with the $\mathcal{L}_1$ metric and this follows from Agaoglu's Theorem. With the same metric, it is possible to show that the mapping T is continuous. Indeed, for any sequence $\{f_j\}_{j \geq 1}$ with $\lim f_j = f$ the corresponding sequences $k_1(f_j)$ and γ_{f_j}

also converge to $k_1(f)$ and γ_f. As the expression of $T(f)$ is also continuous in γ_f, $T(f_j)$ also converges to $T(f)$. Then by Schauder fixed point theorem there is $f \in C([0,1]|g)$ such that $T(f) = f$. □

Now, fixing the behavior of the top cluster, we can concentrate on the remaining population $[0, k_1]$. Similar arguments as in Lemma 9 can be applied here to show the existence of $k_2 \in (0, k_1)$ that satisfies equation (9) and a segment of f on $[k_2, k_1]$ that satisfies the corresponding balance equation (10). ■

Proposition 4 above shows the existence of a search equilibrium outcome but not its uniqueness. It can be possible to have multiple outcomes for the same set of parameters; depending on the underlying population distribution G.

5.2 Platform

When we have a platform, some types will choose to subscribe and others will not. The matching flows and therefore the unmatched type distributions are defined according to the subscription behavior.

For the nonsubscriber types, we denote the search and matching values as $V_o(\cdot)$ and $V_m(\cdot, \cdot)$ and define them as before. The corresponding balance equation also defines $f(\cdot)$ as before. For the subscriber types, we denote the search and matching values as $V_s(\cdot)$ and $V_{ms}(\cdot, \cdot)$.

Now, fix any $\mu_I > 0$ and $\rho > 0$ and a set of subscribers S. For any type a that subscribes to the platform, the search value can be written as

$$V_s(a) = \frac{-\rho + \mu_I \int_{M(a)\cap S} V_{ms}(a,b)dF(x) + \mu_o \int_{M(a)} V_{ms}(a,x)dF(x)}{r + \mu_I \int_{M(a)\cap S} dF(x) + \mu_o \int_{M(a)} dF(x)}, \tag{11}$$

and match value is

$$V_{ms}(a,b) = \frac{rb + \delta V_s(a)}{r + \delta}. \tag{12}$$

The corresponding balance equation for a subscriber type is

$$\delta(g(a) - f(a)) = (\mu_o + \mu_I) f(a) \int_{M(a)} f(x)dx. \tag{13}$$

Participation equilibrium can be defined similarly as before with the only addition of the balance equations.

It is possible to extend Theorem 2 to this set-up and describe the set of participation equilibria with a tree structure. We apply the same selection rule to concentrate on the sequentially maximal participation equilibria. Our first main result in this section is then an extension of Theorem 3, which could be stated as follows:

Theorem 3. *Fix any $\mu_I > 0$ and $\rho > 0$. There exists a sequentially maximal participation equilibrium. This equilibrium can be described by a finite sequence $\{\hat{k}_i\}_{i=0}^n$ for some positive integer n, where $\hat{k}_0 = 1$ and $\hat{k}_i = \min\{k_i, o_i\}$ and a piece-wise continuous distribution function $f(\cdot)$. For each $i \geq 1$ the thresholds k_i and o_i satisfy*

$$V_s^i = \frac{-\rho + (\mu_I + \mu_o)\int_{k_i}^{k_{i-1}} V_{ms}^i(x)dF(x)}{r + (\mu_I + \mu_o)\int_{k_i}^{k_{i-1}} dF(x)} = k_i \quad \& \quad V_o^i = \frac{\mu_o \int_{o_i}^{k_{i-1}} V_m^i(x)dF(x)}{r + \mu_o \int_{o_i}^{k_{i-1}} dF(x)} = o_i, \tag{14}$$

while the unmatched type distribution $f(\cdot)$ satisfies the balance equation (5) over S^c and equation (13) over S.

The set of subscribers is defined as follows: the interval $[k_i, k_{i-1}] \subseteq S$ if and only if $k_i \geq o_i$.

The proof of Theorem 3 follows from a combination of arguments from the proof of Theorems 1 and 2, and the Proposition 4.

5.3 Vanishing Search Frictions

When there is endogenous flow, as we proved in Proposition 3, as the platform's search technology gets indefinitely better, as $\mu_s \to \infty$, the flow of subscribers increase. With higher μ_s, each individual stays subscribed for a shorter period of time in expected terms, but as the flow of subscriber types increase proportionally, the revenue flow of the firm stays proportional to the price. The platform can then increase the price proportionally to μ_s, which yields infinite profit at the limit.

When the flow of types is exogenous, and so the population of unmatched subscriber types vanish as $\mu_s \to \infty$, the revenue flow does not stay proportional to price. Then, it is not immediately clear what happens to the revenue flow to the platform in the limit. Proposition 5 below states that the limiting revenue flow is finite.

Proposition 5. *As search frictions vanish; i.e.* $\mu_s \to \infty$*,*

$$0 < \lim_{\mu_s \to \infty} rV_\pi \leq \lim \left(\sum_{\{i|o_i(\hat{k}_{i-1}) \leq k_i(\hat{k}_{i-1}) < \hat{k}_{i-1}\}} \delta \int_{k_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} xg(x)dx - \left(r + \delta \int_{k_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} g(x)dx \right) \right) \quad (15)$$

Proof Note that for any $\mu_s > 0$, the demand function can be written as

$$D(\rho(\mu_s), \mu_s) = \sum_{i \geq 1|o_i(\hat{k}_{i-1}) \leq k_i(\hat{k}_{i-1})} \int_{k_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx,$$

where $\rho(\mu_s)$ is the optimal price that the platform chooses at μ_s. It is possible that some of the subscriber clusters cease to subscribe and some of the nonsubscriber clusters start to subscribe as the platform adjusts its pricing as a response to μ_s. Therefore, it is sufficient to consider the clusters that are of positive measure in the limit and stay subscribed as $\mu_s \to \infty$. Let $\lim I$ be the index of such clusters. Then

$$\lim I = \{i \geq 1 | 0 < \lim o_i(\hat{k}_{i-1}) \leq \lim k_i(\hat{k}_{i-1}) < \lim \hat{k}_{i-1}\}.$$

Let

$$\lim D(\rho(\mu_s), \mu_s) = \sum_{i \in \lim I} \int_{\lim k_i(\hat{k}_{i-1})}^{\lim \hat{k}_{i-1}} f(x)dx$$

be the limiting demand.

At each μ_s, and $i \in \lim I$ fix $\hat{k}_{i-1}$. Let $\bar{\rho}_i(\hat{k}_{i-1})$ be the price such that $k_i(\hat{k}_{i-1}) = o_i(\hat{k}_{i-1})$. By Proposition 2, $k_i(\hat{k}_{i-1})$ decreases with the price. If the platform could increase price from $\rho(\mu_s)$ without changing the remaining clusters in the society, it would increase its revenue. Therefore, $\rho(\mu_s) \leq \bar{\rho}_i(\hat{k}_{i-1})$. From this inequality, we can infer that

$$\lim \rho(\mu_s) D(\rho(\mu_s), \mu_s) \leq \sum_{i \in \lim I} \lim \bar{\rho}_i \int_{\lim k_i(\hat{k}_{i-1})}^{\lim \hat{k}_{i-1}} f(x)dx.$$

By construction, $\bar{\rho}_i(\hat{k}_{i-1})$ can be written as

$$\bar{\rho}_i(\hat{k}_{i-1}) = (\mu_s + \mu_o) \int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} xf(x)dx - \left(r + (\mu_s + \mu_o) \int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx \right).$$

Now, let's look at the limit of $\bar{\rho}_i(\hat{k}_{i-1}) \int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx)$ as $\mu_s \to \infty$ by employing the following balance equation for any $a \in [o_i(\hat{k}_{i-1}), \hat{k}_{i-1}]$

$$\delta(g(a) - f(a)) = (\mu_s + \mu_o) f(a) \int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx.$$

Note first that if $f(a)$ does not converge to 0, the right-hand side of the balance equation above diverges to ∞, while the left-hand side is always finite. Furthermore, the term $f(a) \int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx$ also converges to 0 and so $\int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx$. Integrating the balance equation above over the interval $[o_i(\hat{k}_{i-1}), \hat{k}_{i-1}]$ yields that

$$\lim(\mu_s + \mu_o) \left(\int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx \right)^2 = \delta \int_{\lim o_i(\hat{k}_{i-1})}^{\lim \hat{k}_{i-1}} g(x)dx \in (0, \infty),$$

while $\lim(\mu_s + \mu_o) \int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx = \infty$. Similarly, multiplying the balance equation with a and then taking the integration yields

$$\lim(\mu_s + \mu_o) \int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} xf(x)dx \int_{o_i(\hat{k}_{i-1})}^{\hat{k}_{i-1}} f(x)dx = \delta \int_{\lim o_i(\hat{k}_{i-1})}^{\lim \hat{k}_{i-1}} xg(x)dx \in (0, \infty)$$

■

A Proofs

Proof of Lemma 1: Given the set of subscribers S, a non-subscriber type $a \in S^c$, where $S^c \equiv [0, 1] \backslash S$, can have matchings from both subscribers and non-subscribers. Let $M(a)$ be matching set of the type a. Then, the value of search for a non-subscriber type a at a fixed time interval Δ can be derived in exactly the same way as in the case with no platform.

For a subscriber $a' \in S$, calculating the value is different in two aspects. First, s/he accounts for the discounted subscription fee they pay. Moreover, the flow rate from the subscribers are different. The search value for a fixed interval of time Δ is

$$V_s(a';\sigma) = -\rho \int_0^{\Delta} e^{-rt}dt + e^{-r\Delta}\left(e^{-\mu_0 \int_{M(a')} dF(x)\Delta} e^{-\mu_s \int_{M(a')\cap S} dF(x)\Delta}\right) V_s(a';\sigma)$$
$$+e^{-r\Delta}\left(\left(e^{-\mu_0 \int_{M(a')} dF(x)\Delta}\right)\mu_s \int_{M(a')\cap S} \Delta \left(e^{-\mu_s \int_{M(a')\cap S} dF(x)\Delta}\right) \frac{\int_{M(a')\cap S} xdF(x)}{\int_{M(a')\cap S} dF(x)}\right)$$
$$+e^{-r\Delta}\left(\left(e^{-\mu_s \int_{M(a')\cap S} dF(x)\Delta}\right)\mu_0 \int_{M(a')} dF(x)\Delta \left(e^{-\mu_0 \int_{M(a')} dF(x)\Delta}\right) \frac{\int_{M(a)} xdF(x)}{\int_{M(a)} dF(x)}\right)$$
$$+\mathcal{O}(\Delta)$$

The expression above simplifies into the following:

$$V_s(a';\sigma) = -\rho\frac{1-e^{-r\Delta}}{r} + \left(e^{-\left(r+\mu_o \int_{M(a')} dF(x)+\mu_s \int_{M(a')\cap S}\right)\Delta}\right) V_s(a';\sigma)$$
$$\left(e^{-\left(r+\mu_o \int_{M(a')} dF(x)+\mu_s \int_{M(a')\cap S}\right)\Delta}\right) \mu_s\Delta \int_{M(a')\cap S} xdF(x)$$
$$\left(e^{-\left(r+\mu_o \int_{M(a')} dF(x)+\mu_s \int_{M(a')\cap S}\right)\Delta}\right) \mu_o\Delta \int_{M(a')} xdF(x) + \frac{\mathcal{O}(\Delta)}{\Delta}$$

Rearranging the terms, dividing all to Δ and letting $\Delta \to 0$ yields the search value in the statement. ■

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